

Function Approximation Technique Based Immersion and Invariance Control for Unknown Nonlinear Systems

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Abstract—A function approximation technique based immersion and invariance (FATII) control method is proposed in this letter. Firstly, an unknown control system is restructured as the combination of an auxiliary system and a variation term from the original system. The variation term is treated as a time-varying uncertainty and parameterized by a group of weighted chosen basis functions. These weights are estimated at every time instant and the change of the estimates is governed by an update law. The update law is defined based on an immersion and invariance approach such that both the system state and the estimation error converge to zero. The FATII method is model-free and thus applicable to a wide range of systems. The asymptotic stability of the proposed method is established and its feasibility is verified under simulations.

Index Terms—Robust adaptive control, uncertain systems.

I. INTRODUCTION

THE NEED for the control of nonlinear systems arises in many practical applications that include the use of ships, underwater vehicles, aircraft, satellites, flexible joint robots, hyper-redundant and snake-like manipulators, walking robots, hybrid machines, etc.

To control these systems, various methods have been proposed in the literature, which can be classified into two main types: model-based and model-free methods. Model-based control methods include energy shaping [1], partial

feedback linearization [2], backstepping [3], sliding mode control [4], [5], etc. They commonly rely upon the model information of the control systems, which limits their range of applications. Also, they are not sufficiently robust to large system uncertainties. Instead of utilizing the model information, a number of model-free methods [6] have been developed for designing controllers directly from the input-output data, bypassing the modelling step. Although they are widely applicable and adaptive to uncertainties, the asymptotic stability of these methods is difficult to prove.

To feature both the stability and the adaptiveness, in this letter, we propose a novel control method (the FATII method) which is model-free, and thus applicable to a wide class of systems and robust to uncertainties. Also, the asymptotic stability of the FATII method can be proved.

The FATII method is based on the function approximation technique (FAT) [7]–[9], inspired by its applications on the adaptive control problems. The FAT-based approaches reconstruct the time-varying uncertainties in the control systems as the combination of a group of weighted basis functions and a remainder term. Then, model reference adaptive control (MRAC) techniques are utilized to deal with the unknown weights, and the effects of the remainder term can be eliminated by a sliding mode controller [10]–[14]. Note that FAT-based designs have a drawback that in the absence of persistent excitation (PE) condition, the estimation for the weights is not guaranteed to converge to the actual value. The drawback would deteriorate the control accuracy and lead to an undesired transient response of the closed loop system [15].

Noticing that the immersion and invariance (I&I) technique [16], [17] can treat the state stabilization and the parameter estimation in a unified manner, for the design of the FATII method, we improve the FAT-based approaches by utilizing the I&I technique instead of the MRAC, such that not only the system state, but also the error between the uncertainty and its estimation, are steered to zero. Nevertheless, the remainder term generated in the function approximation process cannot be eliminated by a sliding mode controller when one directly incorporates the conventional I&I technique into the FAT-based controller. Regarding this problem, we modify the conventional I&I technique with the use of switching in order to deal with the remainder term.

The process for designing FATII controllers is as follows. A control system is reconstructed as the combination of an approximated one and the variation from it. The variation term is then estimated with chosen basis functions weighted by unknown parameters. Update laws are defined based on the

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modified I&I technique such that the parameters can be automatically adjusted and the effect of the uncertainty term to the control process can be eliminated.

The rest of this letter is organized as follows. First, in Section II we state the control problem, illustrate the process for constructing the FATII controller, and establish its asymptotic stability. The feasibility of the proposed control method is verified under simulations in Section III. Finally, conclusions are drawn in Section IV.

II. CONTROLLER DESIGN PROCESS

In this section, the FAT approach is firstly utilized to convert general control systems into a unified form. Next, the FATII control method is proposed, and its stability is established.

A. Statement of Problem

Given a control system written in the state-space form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{G}(\mathbf{x})\mathbf{u} + \boldsymbol{\xi}, \quad (1)$$

where $\mathbf{x} \in \mathbb{R}^n$ represents the state, $\mathbf{u} \in \mathbb{R}^m$ with $n > m$, represents the input, and $\boldsymbol{\xi} \in \mathbb{R}^n$ for the external disturbance which is assumed to be bounded, define an asymptotically stabilizing law \mathbf{u} for system (1).

Assume that $\tilde{\mathbf{f}}$ and $\tilde{\mathbf{G}}$ respectively stand for the nominal drift term and control matrix such that

$$\mathbf{f} = \tilde{\mathbf{f}} + \delta\mathbf{f}, \quad \mathbf{G} = \tilde{\mathbf{G}} + \delta\mathbf{G}, \quad (2)$$

where $\delta\mathbf{f}$ and $\delta\mathbf{G}$ denote the difference between the nominal and actual values. By substituting (2) into (1), one obtains

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}} + \tilde{\mathbf{G}}\mathbf{u} + \tilde{\boldsymbol{\xi}}, \quad (3)$$

where $\tilde{\boldsymbol{\xi}} = \delta\mathbf{f} + \delta\mathbf{G}\mathbf{u} + \boldsymbol{\xi}$ is the lumped error [18]. Note that as stated in [19], the term ‘‘unknown system’’ is utilized in place of ‘‘uncertain system’’ to stress that \mathbf{f} and \mathbf{G} are not known a priori and thus are possibly subject to large uncertainty $\tilde{\boldsymbol{\xi}}$, so that model-based control would not be applicable.

By mimicking the pole placement method for linear systems, one introduces an auxiliary input $\mathbf{u}^* \in \mathbb{R}^m$ to match the dimension of input with that of the output, such that

$$\mathbf{u} = \mathbf{G}^*\mathbf{u}^*, \quad (4)$$

where the auxiliary matrix \mathbf{G}^* is chosen to be a full rank $m \times n$ matrix. Then, the control system (1) is rewritten as

$$\dot{\mathbf{x}} = \tilde{\mathbf{f}} + \tilde{\mathbf{G}}\mathbf{G}^*\mathbf{u}^* + \tilde{\boldsymbol{\xi}}, \quad (5)$$

where the number of the inputs equals that of the states. Note that an essential condition for the selection of matrix \mathbf{G}^* is that the reconstructed system (5) is required to be controllable, otherwise the design of \mathbf{u}^* cannot guarantee the convergence of \mathbf{x} . Methods for the design of \mathbf{G}^* that render system (5) controllable, are specified in the Appendix.

To further simplify (5), one can rearrange it as

$$\dot{\mathbf{x}} = \mathbf{u}^* + \mathbf{d}(\mathbf{x}, t, \mathbf{u}^*), \quad (6)$$

where $\mathbf{d}(\mathbf{x}, t, \mathbf{u}^*) = \tilde{\mathbf{f}} + (\tilde{\mathbf{G}}\mathbf{G}^* - \mathbf{I})\mathbf{u}^* + \tilde{\boldsymbol{\xi}}$. Through the above rearrangement, the original system (1) is restructured as the combination of two parts, a trivial linear system $\dot{\mathbf{x}} = \mathbf{u}^*$ referring to the auxiliary system, and \mathbf{d} , which can be viewed as an uncertainty term to the auxiliary system. Thus the original problem is reformulated to the adaptive control problem for a linear system with time-varying uncertainties, which is stated as designing a locally asymptotically stabilizing law \mathbf{u}^* for (6), with \mathbf{d} unknown.

To tackle the stated problem, we utilize the weighted basis functions to approximate \mathbf{d} in the control system (6) at each time instant as

$$\mathbf{d}(\mathbf{x}, t, \mathbf{u}^*) = \sum_{i=1}^N \mathbf{d}_i \psi_i(\mathbf{x}, t) + \boldsymbol{\epsilon}, \quad (7)$$

where \mathbf{d}_i is constant and ψ_i consists of \mathbf{x} and t , and $\boldsymbol{\epsilon}$, referring to the approximation error, describes the deviation between the uncertainty \mathbf{d} and the weighted basis functions. Substituting (7) into (6) yields

$$\dot{\mathbf{x}} = \mathbf{u}^* + \sum_{i=1}^N \mathbf{d}_i \psi_i(\mathbf{x}, t) + \boldsymbol{\epsilon}. \quad (8)$$

Note that the controllability of the restructured system (5) implies the existence of bounded \mathbf{u}^* for the desired \mathbf{x} such that \mathbf{d} is definable and bounded. Otherwise, the function approximate technique through (7) to (8) is not applicable.

For the control of (8), several remarks are in order.

Remark 1: The control problem requires to eliminate the effect of \mathbf{d}_i in (8). For this purpose, \mathbf{d}_i , at each time instant t , is estimated by $\hat{\mathbf{d}}_i(t)$ through the MRAC techniques in the FAT-based controls. However, $\hat{\mathbf{d}}_i(t)$ is not guaranteed to converge to \mathbf{d}_i in the absence of the PE condition. The error between the actual parameters and the parameter estimation can produce large uncertainty, which would deteriorate the control accuracy and lead to an undesired transient response of the closed-loop system.

A unique feature of the I&I framework is that it allows to treat the state stabilization and the parameter estimation in a unified manner [17]. For the convergence of both the system state and the estimation error, we propose in Section II-B the FATII method based on the I&I approach.

Remark 2: In the FAT-based controller design, the effect of $\boldsymbol{\epsilon}$ in (8) is commonly covered by a sliding mode controller, but this operation cannot be accomplished when the conventional I&I technique is incorporated into the FAT-based controller. Therefore, in the design of the FATII controller, we modify the I&I technique with the use of switching to deal with $\boldsymbol{\epsilon}$.

Remark 3: Several candidates for the basis function ψ_i in (7) can be chosen to approximate the nonlinear functions, and in this letter, we select the Fourier series [20], [21]. From the Weierstrass theorem, when $N \rightarrow \infty$, the approximation error $\boldsymbol{\epsilon}$ would be infinitesimally small. By selecting enough number of the basis functions, the estimation of the uncertainty can be sufficiently accurate such that $\boldsymbol{\epsilon}$ in (10) is negligible [20]. Thus, it is reasonable to have the following [10]–[14]

Assumption 1: The error $\boldsymbol{\epsilon}$ is bounded such that $\|\boldsymbol{\epsilon}\|_1 \leq E_1$ and $\|\boldsymbol{\epsilon}\|_2 \leq E_2$ where E_1 and E_2 are positive constants.

B. FATII Based Controller Design

Define in the extended space $(\mathbf{x}, \hat{\mathbf{d}}_i)$ the manifold

$$\mathcal{M}_i = \{(\mathbf{x}, \hat{\mathbf{d}}_i) \in \mathbb{R}^n \mid \mathbf{d}_i - \hat{\mathbf{d}}_i - \boldsymbol{\beta}_i = 0\}, \quad (9)$$

where $\boldsymbol{\beta}_i(\mathbf{x}, t)$ is a continuous function to be specified. The motivation for this definition is described as follows. The dynamics of (8) restricted to the manifold \mathcal{M}_i (provided it is invariant) is described by

$$\dot{\mathbf{x}} = \mathbf{u}^* + \sum_{i=1}^N (\hat{\mathbf{d}}_i + \boldsymbol{\beta}_i) \psi_i + \boldsymbol{\epsilon}. \quad (10)$$

Note from (10), the unknown vector \mathbf{d}_i is excluded from the expression of $\dot{\mathbf{x}}$, which is important in the controller design process since \mathbf{d}_i cannot appear in the control law.

However, (10) is equivalent to (6) only when the system dynamics stay in the manifold \mathcal{M}_i . By defining the off-the-manifold variable

$$z_i = d_i - \hat{d}_i - \beta_i, \quad (11)$$

where $z_i \in \mathbb{R}^{n \times 1}$ and $\beta_i \in \mathbb{R}^{n \times 1}$, $z_i = \mathbf{0}$ implies that the system dynamics stay in the manifold \mathcal{M}_i . With the off-the-manifold variable, the state equation is transformed to

$$\dot{\mathbf{x}} = \mathbf{u}^* + \sum_{i=1}^N (z_i + \hat{d}_i + \beta_i) \psi_i + \epsilon, \quad (12)$$

where $\sum_{i=1}^N z_i \psi_i$ represents the estimation error of the system uncertainty. Thus, the original control problem can be restated as defining an asymptotically stabilizing law \mathbf{u}^* for both the system state and the estimation error.

Remark 4: As mentioned in Remark 1, by selecting a sufficiently large N , ϵ in (10) is negligible [7], [20]. On the other hand, taking ϵ into account will reduce the number of basis functions and thus, increase the efficiency of the estimation process. To address both cases, in what follows, FATII controllers are proposed respectively when ϵ is not (see Section II-B1) and is negligible (see Section II-B2).

1) *Controller Design When ϵ Is Not Negligible:* When the number of basis function N is selected small (seeking for a low computational load) and thus ϵ is not negligible, define the following controller

$$\begin{aligned} \mathbf{u}^* &= -\mathbf{K}\mathbf{x} - \gamma \text{sgn}(\mathbf{x}) - \sum_{i=1}^N (\hat{d}_i + \beta_i) \psi_i, \\ \dot{\hat{d}}_i &= \begin{cases} -\mathbf{x} \dot{\psi}_i + (\mathbf{K}\mathbf{x} + \gamma \text{sgn}(\mathbf{x})) \psi_i, & \|\mathbf{x}\|_2 > \delta \\ \mathbf{x} \psi_i, & \|\mathbf{x}\|_2 \leq \delta \end{cases} \\ \beta_i &= \begin{cases} \mathbf{x} \psi_i, & \|\mathbf{x}\|_2 > \delta \\ 0, & \|\mathbf{x}\|_2 \leq \delta \end{cases} \end{aligned} \quad (13)$$

where δ is an arbitrary positive constant, $\gamma > E_1$, $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{I}$, and \mathbf{K}_1 , \mathbf{K}_2 are positive definite matrices satisfying that $\lambda_{\min}(\mathbf{K}_2) \geq E_2^2/\delta^2$.

Remark 5: It can be seen that the closed-loop system formulated by the plant (12) and controller (13) is a switched nonsmooth system. Thus, the traditional Lyapunov techniques are not applicable. To address this problem, the generalized Lasalle-Yoshizawa theorem in [22] is employed, where the main idea is stated as follows: one can establish asymptotic properties for the generalized solutions of the switched nonsmooth system by using those of its subsystems.

In addition, for a differential equation $\dot{\mathbf{x}} = \mathbf{h}(\mathbf{x}, t)$ with discontinuous right-hand side, an absolutely continuous function $\mathbf{x}(t)$ is called a generalized solution (Filippov solution) to it on $[t_0, t_1]$ if $\dot{\mathbf{x}} \in \mathbf{F}[\mathbf{h}](\mathbf{x}, t)$, where

$$\mathbf{F}[\mathbf{h}](\mathbf{x}, t) = \bigcap_{\rho > 0} \bigcap_{\mu(\mathcal{N})=0} \overline{\text{co}} \mathbf{h}(\mathbf{B}(\mathbf{x}, \rho) - \mathcal{N}, t), \quad (14)$$

represents the Filippov regularization [23]. In (14), $\overline{\text{co}}$ denotes the convex closure, $\mathbf{B}(\mathbf{x}, \rho)$ the open ball of radius ρ centered at \mathbf{x} , and $\bigcap_{\mu(\mathcal{N})=0}$ the intersection over all sets \mathcal{N} of Lebesgue measure zero. It has the following property [24]

$$\mathbf{F}[\mathbf{h}_1 + \mathbf{h}_2](\mathbf{x}, t) \subseteq \mathbf{F}[\mathbf{h}_1](\mathbf{x}, t) + \mathbf{F}[\mathbf{h}_2](\mathbf{x}, t), \quad (15)$$

which will be used in the following.

Theorem 1: Every maximal solution of the Filippov regularization of the switched nonsmooth system formulated by (12) and (13), regardless of the initial condition, is complete [25], bounded, and satisfies $\lim_{t \rightarrow \infty} \mathbf{x}(t) = \mathbf{0}$.

Proof: Define the state-dependent switching signal $\sigma(\mathbf{x})$, where $\sigma = 1$ represents $\|\mathbf{x}\|_2 > \delta$, and $\sigma = 2$ denotes $\|\mathbf{x}\|_2 \leq \delta$. The following steps are based on the notation of the switching signals.

By substituting (13) into (12), one obtains

$$\dot{\mathbf{x}} = -\mathbf{K}\mathbf{x} - \gamma \text{sgn}(\mathbf{x}) + \sum_{i=1}^N z_i \psi_i + \epsilon, \quad (16)$$

and the substitution of (16) into the derivative of (11) gives

$$\dot{z}_i = \mathbf{g}_{i,\sigma} = \begin{cases} -\psi_i \left(\sum_{j=1}^N z_j \psi_j + \epsilon \right), & \sigma = 1 \\ -\mathbf{x} \psi_i, & \sigma = 2 \end{cases} \quad (17)$$

By introducing $\mathbf{y} = [\mathbf{x}^\top \ z_1^\top \ \dots \ z_N^\top]^\top$, (16), (17) can be written in the augmented state space form as $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$, the subsystem of which is described as

$$\dot{\mathbf{y}} = \mathbf{f}_\sigma(\mathbf{y}, t), \quad \sigma = 1, 2 \quad (18)$$

where

$$\mathbf{f}_\sigma(\mathbf{y}, t) = \begin{bmatrix} -\mathbf{K}\mathbf{x} - \gamma \text{sgn}(\mathbf{x}) + \sum_{i=1}^N z_i \psi_i + \epsilon \\ \mathbf{g}_{1,\sigma}(\mathbf{y}, t) \\ \vdots \\ \mathbf{g}_{N,\sigma}(\mathbf{y}, t) \end{bmatrix}. \quad (19)$$

According to (14) and (15), define $\mathbf{F}_\sigma(\mathbf{x}, t) = \mathbf{F}[\mathbf{f}_\sigma](\mathbf{x}, t)$ as the Filippov regularization of the subsystem (18), satisfying that $\mathbf{F}_\sigma(\mathbf{y}, t) \subseteq \mathbf{F}'_\sigma(\mathbf{y}, t)$, where

$$\mathbf{F}'_\sigma(\mathbf{y}, t) = \begin{bmatrix} \{-\mathbf{K}\mathbf{x} + \sum_{i=1}^N z_i \psi_i + \epsilon\} - \gamma \mathbf{F}[\text{sgn}](\mathbf{x}) \\ \{\mathbf{g}_{1,\sigma}\} \\ \vdots \\ \{\mathbf{g}_{N,\sigma}\} \end{bmatrix}.$$

To study the generalized solutions of the closed-loop system formulated by (12) and (13) based on its afore-generated subsystems, one constructs the following Lyapunov candidate function

$$V = \frac{1}{2} \mathbf{y}^\top \mathbf{y} = \frac{1}{2} \mathbf{x}^\top \mathbf{x} + \frac{1}{2} \sum_{i=1}^N z_i^\top z_i. \quad (20)$$

Since V , expressed by (20), is smooth, the Clarke gradient is reduced to the standard gradient, as $\partial V = \{\mathbf{y}\}$. It should be noted that $\mathbf{x}^\top \mathbf{F}[\text{sgn}](\mathbf{x}) = \{\|\mathbf{x}\|_1\}$ [22]. Thus, a bound on the generalized time derivative of the Lyapunov candidate function $\overline{V}_\sigma(\sigma = 1, 2)$ can be expressed as

$$\begin{aligned} \dot{\overline{V}}_\sigma &= \max_{q \in \mathbf{F}'_\sigma(\mathbf{y}, t)} \mathbf{y}^\top \mathbf{q} \\ &\leq \max_{q \in \mathbf{F}'_\sigma(\mathbf{y}, t)} \mathbf{y}^\top \mathbf{q} = \mathbf{x}^\top \left(-\mathbf{K}\mathbf{x} + \sum_{i=1}^N z_i \psi_i + \epsilon \right) \\ &\quad - \gamma \|\mathbf{x}\|_1 + \sum_{i=1}^N z_i^\top \mathbf{g}_{i,\sigma}. \end{aligned} \quad (21)$$

For both $\sigma = 1$ and $\sigma = 2$, we need to analyze whether the corresponding $\dot{\overline{V}}_\sigma$ is negative semidefinite.

When $\sigma = 1$, by substituting the expression of $\mathbf{g}_{i,\sigma}$ as (17) into (21), one obtains

$$\dot{\overline{V}}_\sigma \leq \mathbf{x}^\top \left(-\mathbf{K}\mathbf{x} + \sum_{i=1}^N z_i \psi_i + \epsilon \right)$$

$$- \left(\sum_{i=1}^N z_i \psi_i \right)^\top \left(\sum_{i=1}^N z_i \psi_i + \epsilon \right) - \gamma \|\mathbf{x}\|_1,$$

a rearrangement of which gives

$$\begin{aligned} \dot{\bar{V}}_\sigma &\leq -\mathbf{x}^\top (\mathbf{K}_1 + \mathbf{K}_2) \mathbf{x} - \gamma \|\mathbf{x}\|_1 + \epsilon^\top \epsilon \\ &\quad - \frac{1}{2} \left(\left(\sum_{i=1}^N z_i \psi_i \right)^\top \left(\sum_{i=1}^N z_i \psi_i \right) + 2 \left(\sum_{i=1}^N z_i \psi_i \right)^\top \epsilon + \epsilon^\top \epsilon \right) \\ &\quad - \frac{1}{2} \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{x}^\top \sum_{i=1}^N z_i \psi_i + \left(\sum_{i=1}^N z_i \psi_i \right)^\top \left(\sum_{i=1}^N z_i \psi_i \right) \right) \\ &\quad - \frac{1}{2} \left(\mathbf{x}^\top \mathbf{x} - 2\mathbf{x}^\top \epsilon + \epsilon^\top \epsilon \right), \end{aligned} \quad (22)$$

where $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{I}$ and $\mathbf{K}_1, \mathbf{K}_2$ are positive-definite matrices. One can further simplify (22) as

$$\begin{aligned} \dot{\bar{V}}_\sigma &\leq -\mathbf{x}^\top (\mathbf{K}_1 + \mathbf{K}_2) \mathbf{x} + \|\epsilon\|_2^2 - \gamma \|\mathbf{x}\|_1 - \frac{1}{2} \|\mathbf{x} - \epsilon\|_2^2 \\ &\quad - \frac{1}{2} \left\| \sum_{i=1}^N z_i \psi_i - \mathbf{x} \right\|_2^2 - \frac{1}{2} \left\| \sum_{i=1}^N z_i \psi_i + \epsilon \right\|_2^2 \\ &\leq -\mathbf{x}^\top (\mathbf{K}_1 + \mathbf{K}_2) \mathbf{x} + \|\epsilon\|_2^2 \\ &\leq -\lambda_{\min}(\mathbf{K}_1) \|\mathbf{x}\|_2^2 - \lambda_{\min}(\mathbf{K}_2) \|\mathbf{x}\|_2^2 + E_2^2. \end{aligned} \quad (23)$$

As $\lambda_1 = \lambda_{\min}(\mathbf{K}_1) > 0$, $\lambda_{\min}(\mathbf{K}_2) \geq E_2^2/\delta^2$, and $\sigma(\mathbf{x}) = 1$ ($\|\mathbf{x}\|_2 > \delta$), one obtains $\dot{\bar{V}}_\sigma \leq -\lambda_1 \|\mathbf{x}\|_2^2$.

When $\sigma = 2$, by substituting (17) into (21), the generalized time derivative of the Lyapunov candidate function $\dot{\bar{V}}_\sigma$ is transformed into

$$\begin{aligned} \dot{\bar{V}}_\sigma &\leq \mathbf{x}^\top \left(-\mathbf{K}\mathbf{x} + \sum_{i=1}^N z_i \psi_i + \epsilon \right) - \gamma \|\mathbf{x}\|_1 - \sum_{i=1}^N z_i^\top \mathbf{x} \psi_i \\ &\leq -\mathbf{x}^\top \mathbf{K}\mathbf{x} + \mathbf{x}^\top \epsilon - \gamma \|\mathbf{x}\|_1. \end{aligned} \quad (24)$$

The Cauchy-Schwarz inequality gives $\mathbf{x}^\top \epsilon \leq \|\mathbf{x}\|_2 \|\epsilon\|_2$, and as $\|\cdot\|_2 \leq \|\cdot\|_1$, $\mathbf{x}^\top \epsilon \leq \|\mathbf{x}\|_1 \|\epsilon\|_1$. Since $\mathbf{K} = \mathbf{K}_1 + \mathbf{K}_2 + \mathbf{I}$, one obtains

$$\dot{\bar{V}}_\sigma \leq -\mathbf{x}^\top \mathbf{K}_1 \mathbf{x} - (\gamma - \|\epsilon\|_1) \|\mathbf{x}\|_1 \leq -\lambda_1 \|\mathbf{x}\|_2^2 - (\gamma - \|\epsilon\|_1) \|\mathbf{x}\|_1.$$

As $\|\epsilon\|_1 \leq E_1$, by selecting $\gamma > E_1$, $\dot{\bar{V}}_\sigma \leq -\lambda_1 \|\mathbf{x}\|_2^2$ when $\sigma = 2$.

Thus, in conclusion, for both cases when $\sigma = 1$ and $\sigma = 2$,

$$\dot{\bar{V}}_\sigma \leq -\lambda_1 \|\mathbf{x}\|_2^2. \quad (25)$$

According to the generalized LaSalle-Yoshizawa Theorem [22, Th. 3], for switched nonsmooth systems with finite number of subsystems being active in a small neighborhood of the state space, all maximal generalized solutions of the switched nonsmooth system formulated by (16) and (17) are complete [25], bounded, and satisfy $\lim_{t \rightarrow \infty} \mathbf{x} = \mathbf{0}$. ■

2) Controller Design When ϵ Is Negligible: When there are a sufficient number of basis functions, ϵ is negligible, and thus, the plant can be simplified from (12) into

$$\dot{\mathbf{x}} = \mathbf{u}^* + \sum_{i=1}^N (z_i + \hat{d}_i + \beta_i) \psi_i. \quad (26)$$

The controller is designed as

$$\mathbf{u}^* = -\mathbf{K}\mathbf{x} - \sum_{i=1}^N (\hat{d}_i + \beta_i) \psi_i,$$

$$\begin{aligned} \dot{\hat{d}}_i &= -\mathbf{x} \dot{\psi}_i + \mathbf{K}\mathbf{x} \psi_i, \\ \beta_i &= \mathbf{x} \psi_i, \end{aligned} \quad (27)$$

where \mathbf{K} is a positive definite matrix satisfying $\lambda_{\min}(\mathbf{K}) > \frac{1}{4}$. The stability of the proposed controller (27) is proved in the following.

Theorem 2: The closed loop system, formulated by (10) and (27), is asymptotically stable, and the uncertainty estimation converges to the actual uncertainty asymptotically.

Proof: The substitution of (27) into (26) yields

$$\dot{\mathbf{x}} = -\mathbf{K}\mathbf{x} + \sum_{i=1}^N z_i \psi_i. \quad (28)$$

To prove the stability of the closed-loop system, the Lyapunov candidate function can be formulated as

$$V = \frac{1}{2} \left(\mathbf{x}^\top \mathbf{x} + \sum_{i=1}^N z_i^\top z_i \right), \quad (29)$$

the derivative of which is calculated as

$$\dot{V} = \mathbf{x}^\top \dot{\mathbf{x}} + \sum_{i=1}^N z_i^\top \dot{z}_i. \quad (30)$$

The derivative of z_i is computed as

$$\begin{aligned} \dot{z}_i &= -\dot{\hat{d}}_i - \frac{\partial \beta_i}{\partial \mathbf{x}} \dot{\mathbf{x}} - \frac{\partial \beta_i}{\partial t} \\ &= \mathbf{x} \dot{\psi}_i - \mathbf{K}\mathbf{x} \psi_i - \mathbf{x} \dot{\psi}_i - \psi_i \left(-\mathbf{K}\mathbf{x} + \sum_{j=1}^N z_j \psi_j \right) \\ &= -\psi_i \left(\sum_{j=1}^N z_j \psi_j \right). \end{aligned} \quad (31)$$

By substituting (28) and (31) into (30), one obtains

$$\begin{aligned} \dot{V} &= -\mathbf{x}^\top \left(\mathbf{K}\mathbf{x} - \sum_{i=1}^N z_i \psi_i \right) - \left(\sum_{i=1}^N z_i \psi_i \right)^\top \left(\sum_{i=1}^N z_i \psi_i \right) \\ &= -\mathbf{x}^\top \mathbf{K}\mathbf{x} + \frac{1}{4} \mathbf{x}^\top \mathbf{x} \\ &\quad - \left(\sum_{i=1}^N z_i \psi_i - \frac{1}{2} \mathbf{x} \right)^\top \left(\sum_{i=1}^N z_i \psi_i - \frac{1}{2} \mathbf{x} \right) \\ &\leq -\left(\lambda_{\min}(\mathbf{K}) - \frac{1}{4} \right) \|\mathbf{x}\|_2^2 - \left\| \sum_{i=1}^N z_i \psi_i - \frac{1}{2} \mathbf{x} \right\|_2^2, \end{aligned} \quad (32)$$

where $\|\cdot\|_2$ represents the l_2 norm for a vector.

Since $\lambda_{\min}(\mathbf{K}) > \frac{1}{4}$, \dot{V} is negative semi-definite, and therefore, \mathbf{x}, z_i are bounded. According to (28) and (31), as all components of $\dot{\mathbf{x}}$ and \dot{z}_i are bounded, so are $\dot{\mathbf{x}}$ and \dot{z}_i . Moreover, it can be obtained from (32) that

$$\begin{aligned} \ddot{V} &= -2\mathbf{x}^\top \mathbf{K}\dot{\mathbf{x}} + \frac{1}{2} \mathbf{x}^\top \dot{\mathbf{x}} \\ &\quad - 2 \left(\sum_{i=1}^N z_i \psi_i - \frac{1}{2} \mathbf{x} \right)^\top \left(\sum_{i=1}^N (\dot{z}_i \psi_i + z_i \dot{\psi}_i) - \frac{1}{2} \dot{\mathbf{x}} \right), \end{aligned}$$

the components of which are proved bounded. Therefore, so is \ddot{V} , implying that \dot{V} is uniformly continuous. Thus, one

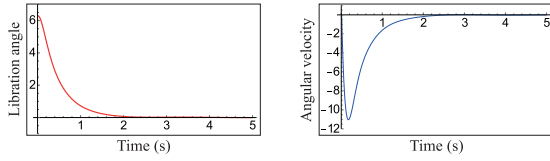


Fig. 1. Trajectories for the libration angle (left) and the angular velocity (right) of a satellite under the FATII control.

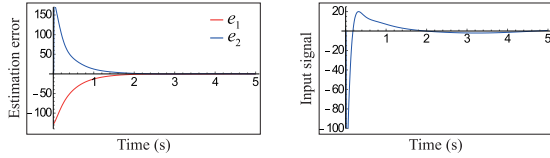


Fig. 2. Uncertainty estimation (left) and the input signals (right) under the FATII control.

obtains $\dot{V} \rightarrow 0$, indicating

$$\lim_{t \rightarrow \infty} \mathbf{x} = \mathbf{0}, \quad \lim_{t \rightarrow \infty} \sum_{i=1}^N \mathbf{d}_i \psi_i = \lim_{t \rightarrow \infty} \sum_{i=1}^N (\hat{\mathbf{d}}_i + \boldsymbol{\beta}_i) \psi_i, \quad (33)$$

according to the Barbalat's lemma [26, Lemma 4.3]. ■

III. CASE STUDY

In this section, we test the validity of the constructed FATII controllers applied on a chaotic system and a nonholonomic system. Note that the robustness of the proposed controllers are also demonstrated in the simulations based on the fact that the main portions of systems are treated as time-varying uncertainties.

A. Chaotic System

Dealing with chaotic motions of the libration angle of a satellite is a significant research topic in the control of spacecrafts

The satellite system can be modeled as [27]

$$C\ddot{\phi} + c\dot{\phi} + \zeta(\phi, t) = M_c, \quad (34)$$

where $\zeta(\phi, t) = 3\omega^2(B - A) \sin \phi \cos \phi + \mu_m i l r^{-3} (2 \sin \phi \sin \omega_c t + \cos \phi \cos \omega_c t)$, and ϕ is the libration angle, M_c the input, $c, \omega_c, A, B, l_m, I, r, i$ the constant parameters.

The motion equation (34) can be written in the state space form as $\dot{\mathbf{x}} = \mathbf{f} + \mathbf{g}u$ where $\mathbf{x} = (\phi, \dot{\phi})$ is the state vector and $u = M_c$ the input. The gain matrix \mathbf{K} in the controller (27) is selected as $\text{diag}(20, 10)$ and matrix \mathbf{G}^* as $[1, 1]$ such that (A.2) is satisfied (see the Appendix). The values of the constant parameters in the model are selected the same as in [27]. Note that $\zeta(\phi, t)$ can be viewed as the system uncertainty which does not appear in the controller.

The initial conditions of the system state ϕ and $\dot{\phi}$ are specified as π rad, π rad/s. The estimation error ($e_1(t), e_2(t)$) = $\mathbf{d} - \sum_{i=1}^N \mathbf{d}_i \psi_i$ between the auxiliary and the actual systems converge to zero, as illustrated in Fig. 2, where $i = 1, 2$. The input signals are also shown by Fig. 2. Note that most works in the literature suppressed the vibration of the libration angle while the FATII method completely eliminates it.

Note that a sliding mode control based strategy employing recursive least squares for nonlinear terms estimation [4], [5] can also be used for this system, whereas the PE condition is required to guarantee that $\hat{\mathbf{d}}$ converges to the actual uncertainty. In comparison, the PE condition is dispensable for the

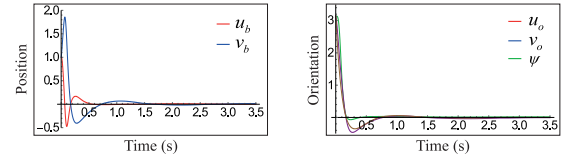


Fig. 3. Trajectories for the position (left) and the orientation (right) of a rolling ball under the FATII control.

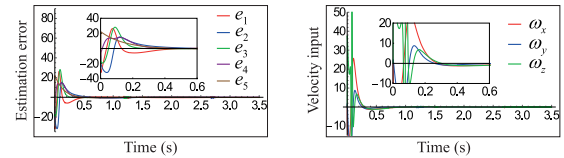


Fig. 4. Uncertainty estimation (left) and the input signals (right) under the FATII control.

proposed controller. However, the controller in [4], [5] can deal with the situation that time-delay is included in the output signal.

B. Nonholonomic System

The control of nonholonomic systems is difficult because, according to the Brockett's condition [28], there exists no continuous state feedback asymptotically stabilizing such type of systems at the equilibrium.

To demonstrate the feasibility of the proposed FATII controller on nonholonomic systems, a rolling ball on a plane is selected. The kinematics of the rolling system is described by Montana's equations [29], [30], as $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})\mathbf{u}$, where $\mathbf{x} = (u_b, v_b, u_o, v_o, \psi)$ and $\mathbf{u} = (\omega_x, \omega_y, \omega_z)$ define respectively the configuration and the angular velocity of the rolling ball, and

$$\mathbf{G}(\mathbf{x}) = \begin{bmatrix} 0 & R & 0 \\ -R & 0 & 0 \\ -\sin \psi / \cos v_o & -\cos \psi / \cos v_o & 0 \\ -\cos \psi & \sin \psi & 0 \\ -\sin \psi \tan v_o & -\cos \psi \tan v_o & -1 \end{bmatrix}. \quad (35)$$

In the simulation, the gain matrix \mathbf{K} in the controller (13) is selected as $\text{diag}(30, 10, 6, 5, 5)$. The trajectories for the configuration of the rolling ball, including its position $u_b(t), v_b(t)$, and its orientation $u_o(t), v_o(t), \psi(t)$, converge to zero, as illustrated in Fig. 3. The initial conditions of the system state $u_b(t), v_b(t), u_o(t), v_o(t)$, and $\psi(t)$ are specified as 1m, 1m, π rad, π rad, and π rad. The estimation errors between the auxiliary and the actual systems converge to zero, as illustrated in Fig. 4. The initial values of the control parameters $\hat{\mathbf{d}}_i(t)$ are chosen to be zero. The input signals are also shown by Fig. 4.

IV. CONCLUSION

The FATII control method has been proposed for a wide class of nonlinear systems. The FATII method has the following features. Firstly, it is model-free and thus, is applicable to a wide range of systems. Secondly, the design of the FATII controller is based on a robust adaptive approach (FAT) and therefore can reject the effect of the system uncertainties or external disturbances to the control system. Thirdly, unlike other model-free methods such as the soft computing techniques, the stability for systems under the FATII control has been well established.

The following issues need to be clarified in the future work. Firstly, a unified way of selecting \mathbf{G}^* needs to be developed. Secondly, we chose the Fourier functions to for estimating the variation term \mathbf{d} . However, the advantages and disadvantages for this type of basis function are not analyzed and more candidates need to be investigated. Thirdly, the delays in the output signal need to be taken into account. Also, more systems will be tested under the FATII based control and experimental works will be conducted.

APPENDIX

It should be noted that there is no unique way for defining \mathbf{G}^* in (4). Depending on the estimation of control systems by (3), it can be selected based on the following considerations.

Firstly, for the estimated control system (3) whose tangent linearization preserves controllability, one can design \mathbf{G}^* as follows. The linearization of (3) at the equilibrium $(\mathbf{x}_e, \mathbf{u}_e)$, without the consideration of ξ , gives

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad (\text{A.1})$$

where $\mathbf{A} = \frac{\partial(\tilde{\mathbf{f}} + \tilde{\mathbf{G}}\mathbf{u})}{\partial\mathbf{x}}|_{\mathbf{x}_e, \mathbf{u}_e}$ and $\mathbf{B} = \frac{\partial(\tilde{\mathbf{f}} + \tilde{\mathbf{G}}\mathbf{u})}{\partial\mathbf{u}}|_{\mathbf{x}_e, \mathbf{u}_e}$. One selects \mathbf{G}^* in (4) as a constant matrix (see Section III-A) such that

$$\text{Re}[\lambda(\mathbf{A}_{n \times n} - \mathbf{B}_{n \times m}\mathbf{G}_{m \times n}^*\mathbf{K}_{n \times n})] < 0, \quad (\text{A.2})$$

where \mathbf{K} is positive definite. Thus, the closed loop system

$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{G}^*\mathbf{K})\mathbf{x}, \quad (\text{A.3})$$

formulated by the linearized system (A.1) and the state feedback portion $\mathbf{u} = -\mathbf{G}^*\mathbf{K}\mathbf{x}$ of the FATII controller, is stable. The effect of the variation between the linearized system (A.1) and the original system (1) can be eliminated by the rest part of the FATII controller. Note that the stability of (A.3) also indicates the controllability of a linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{G}^*\mathbf{u}^*, \quad (\text{A.4})$$

where $\mathbf{u}^* \in \mathbb{R}^n$ is viewed as the input. It is because the selection of $\mathbf{u}^* = -\mathbf{K}\mathbf{x}$ renders (A.4) to the stable form (A.3). As (A.4) is the linearized system of (5) at the equilibrium, the controllability of (A.4) implies the local controllability of the restructured system (5) at the equilibrium [31].

Secondly, for control systems whose tangent linearization does not preserve controllability, such as the nonholonomic systems (see Section III-B), one may select matrix \mathbf{G}^* as the weighted pseudoinverse of $\tilde{\mathbf{G}}$ as $\mathbf{G}^* = (\tilde{\mathbf{G}}^\top \mathbf{W} \tilde{\mathbf{G}})^{-1} \tilde{\mathbf{G}}^\top \mathbf{W}$. The constant matrix \mathbf{W} is designed such that system (5) is controllable. The controllability proof of the restructured system (5) for typical nonholonomic systems such as the unicycle system and the spherical rolling robot can be found in [30]. An alternative way is to select the nilpotent approximation for nonholonomic systems as the nominal plant, and then design \mathbf{G}^* through the process from (A.1) to (A.2).

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